

An intermediate regime for exit phenomena driven by non-Gaussian Lévy noises*

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Abstract

A dynamical system driven by non-Gaussian Lévy noises of small intensity is considered. The first exit time of solution orbits from a bounded neighborhood of an attracting equilibrium state is estimated. For a class of non-Gaussian Lévy noises, it is shown that the mean exit time is asymptotically faster than exponential (the well-known Gaussian Brownian noise case) but slower than polynomial (the stable Lévy noise case), in terms of the reciprocal of the small noise intensity.

Key Words: Stochastic dynamical systems; non-Gaussian Lévy processes; Lévy jump measure; First exit time; Small noise limit

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Dedicated to Professor Ludwig Arnold on the occasion of his 70th birthday

1 Introduction

Although Gaussian processes like Brownian motion have been widely used in modeling fluctuations in engineering and science, it turns out that some complex phenomena involve with non-Gaussian Lévy motions. For instance, it has been argued that diffusion by geophysical turbulence [13] corresponds,

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loosely speaking, to a series of “pauses”, when the particle is trapped by a coherent structure, and “flights” or “jumps” or other extreme events, when the particle moves in the jet flow. Paleoclimatic data [4] also indicates such irregular processes.

Lévy motions are thought to be appropriate models for non-Gaussian processes with jumps [11]. Let us recall that a Lévy motion $L(t)$, or L_t , has independent and stationary increments, i.e., increments $\Delta L(t, \Delta t) = L(t + \Delta t) - L(t)$ are stationary (therefore ΔL has no statistical dependence on t) and independent for any non overlapping time lags Δt . Moreover, its sample paths are only continuous in probability, namely, $\mathbb{P}(|L(t) - L(t_0)| \geq \delta) \rightarrow 0$ as $t \rightarrow t_0$ for any positive δ . This continuity is weaker than the usual continuity in time.

This generalizes the Brownian motion $B(t)$, as $B(t)$ satisfies all these three conditions. But *Additionally*, (i) Almost every sample path of the Brownian motion is continuous in time in the usual sense and (ii) Brownian motion’s increments are Gaussian distributed.

SDEs driven by non-Gaussian Lévy noises have attracted much attention recently [1, 12]. Although the SDEs driven by Lévy motion may generate stochastic flows [9, 1], or generate random dynamical systems in the sense of Arnold [2], under certain conditions, this research issue is still under development. Recently, mean exit time estimates have been investigated by Imkeller and Pavlyukevich [7, 8].

Consider a scalar deterministic ordinary differential equation $\dot{Y}_t = -U'(Y_t)$, $Y_0 = x \in [-b, a]$, $a, b > 0$, where the potential function U is a sufficiently smooth function. Assume that 0 is a asymptotically stable equilibrium. Namely, for any starting point x in $[-b, a]$, the trajectory Y_t tends to 0 as time $t \rightarrow \infty$. In this case, $U(\cdot)$ has a minimum at 0.

Now perturb the deterministic dynamical system Y_t with some small random noise. Let us consider a scalar stochastic differential equation (SDE)

$$dX_t^\varepsilon = -U'(X_t^\varepsilon)dt + \varepsilon dL_t, \quad X_0 = x, \quad (1)$$

where $0 < \varepsilon \ll 1$ is the noise intensity, and L_t is a Lévy process. A scalar Lévy process is characterized by a drift parameter θ , a variance parameter $d > 0$ and a non-negative Borel measure ν , defined on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and concentrated on $\mathbb{R} \setminus \{0\}$, which satisfies

$$\int_{\mathbb{R} \setminus \{0\}} (y^2 \wedge 1) \nu(dy) < \infty, \quad (2)$$

or equivalently

$$\int_{\mathbb{R} \setminus \{0\}} \frac{y^2}{1 + y^2} \nu(dy) < \infty. \quad (3)$$

This measure ν is the so called Lévy measure or the Lévy jump measure of the Lévy process $L(t)$. We also call (θ, d, ν) the *generating triplet*.

We study the first exit problem for the solution process X_t^ε from bounded intervals containing the attracting equilibrium 0, as $\varepsilon \downarrow 0$.

We define the first exit time from the spatial interval $[-b, a]$, a and b positive, as follows:

$$\sigma(\varepsilon) = \inf\{t \geq 0, X_t^\varepsilon \notin [-b, a]\}.$$

It is known [5] that when the noise is Gaussian Brownian, i.e., the Lévy measure part is absent, the mean exit time of the perturbed system X_t^ε is asymptotically exponentially fast

$$E_x \sigma(\varepsilon) \sim \exp\left(\frac{C}{\varepsilon^2}\right) \quad (4)$$

for some positive constant C . Note that here and hereafter E_x is the expectation with respect to the probability law of X_t starting at $X_0 = x$.

If the Lévy measure part is not absent, the exit problem has recently been studied. For symmetric α -stable Lévy noise, i.e., the Lévy process whose Lévy jump measure is $\nu(dy) = \frac{dy}{|y|^{1+\alpha}}$ with $0 < \alpha < 2$, Imkeller and Pavlyukevich [7] have shown that the mean exit time is polynomially fast, $O(\frac{1}{\varepsilon^\alpha})$, in terms of $\frac{1}{\varepsilon}$. Namely, there exist positive constants ε_0 , γ and $\delta > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$,

$$E_x \sigma(\varepsilon) \sim \frac{\alpha}{\varepsilon^\alpha} \left[\frac{1}{a^\alpha} + \frac{1}{b^\alpha} \right]^{-1} (1 + O(\varepsilon^\delta)). \quad (5)$$

for any $x \in [-b + \gamma, a - \gamma]$.

Furthermore, for a class of Lévy noise of exponentially light jumps, Imkeller, Pavlyukevich and Wetzel [8] have shown that the mean exit time is exponentially fast, in terms of $\frac{1}{\varepsilon}$, namely, $O(\exp(\frac{1}{\varepsilon^\alpha}))$ with $\alpha \in (0, 1)$, or $O(\exp(\frac{|\ln \varepsilon|^{1-\frac{1}{\alpha}}}{\varepsilon}))$ with $\alpha > 1$.

In this paper, for a class of non-Gaussian Lévy noises, we show that the mean exit time is asymptotically $O(\frac{|\ln \varepsilon|}{\varepsilon^\alpha})$. This is faster than exponential (the well-known Gaussian Brownian noise case) but slower than the polynomial (the stable Lévy noise case). So we have an intermediate regime,

$$O\left(\frac{1}{\varepsilon^\alpha}\right) < O\left(\frac{|\ln \varepsilon|}{\varepsilon^\alpha}\right) < \exp\left(\frac{C}{\varepsilon^2}\right), \quad (6)$$

for $0 < \varepsilon \ll 1$.

In section 2, we recall the generators for Lévy processes, and then prove the main result. In section 3, we consider two examples of SDEs driven by symmetric Lévy noises, including the α -stable symmetric Lévy noises.

2 Main results

Let L_t be a Lévy process with the generating triplet (θ, d, ν) .

It is known that any Lévy process is completely determined by the Lévy-Khintchine formula (See [1, 11, 10]). This says that for any one-dimensional Lévy process L_t , there exists a $\theta \in \mathbb{R}$, $d > 0$ and a measure ν such that

$$Ee^{i\lambda L_t} = \exp\{i\theta\lambda t - dt\frac{\lambda^2}{2} + t \int_{\mathbb{R}\setminus\{0\}} (e^{i\lambda y} - 1 - i\lambda y I\{|y| < 1\})\nu(dy)\}, \quad (7)$$

where $I(S)$ is the indicator function of the set S , i.e., it takes value 1 on this set and takes zero value otherwise.

The generator A of the process L_t is the same as infinitesimal generator since Lévy process has independent and stationary increments. Hence A is defined as $A\varphi = \lim_{t \downarrow 0} \frac{P_t\varphi - \varphi}{t}$ where $P_t\varphi(x) = E_x\varphi(L_t)$ and φ is any function belonging to the domain of the operator A . Recall the generator A for L_t is (See [1, 10])

$$A\varphi(x) = a\varphi'(x) + \frac{1}{2}d\varphi''(x) + \int_{\mathbb{R}\setminus\{0\}} [\varphi(x+y) - \varphi(x) - I\{|y| < 1\} y\varphi'(x)] \nu(dy). \quad (8)$$

Let us find out the generator of εL_t .

Lemma 1. *Let L_t be a Lévy process with the generating triplet (a, d, ν) . Then for any $\varepsilon > 0$, the generator for εL_t is*

$$A^\varepsilon\varphi = \varepsilon a\varphi'(x) + \frac{1}{2}\varepsilon^2 d\varphi''(x) + \int_{\mathbb{R}\setminus\{0\}} [\varphi(x+\varepsilon y) - \varphi(x) - \varepsilon I\{|y| < 1\} y\varphi'(x)] \nu(dy). \quad (9)$$

Proof. Notice that

$$\begin{aligned} Ee^{i\lambda\varepsilon L_1} &= \exp\{ia\varepsilon\lambda - d\varepsilon^2\frac{\lambda^2}{2} + \int_{\mathbb{R}\setminus\{0\}} (e^{i\lambda\varepsilon y} - 1 - i\varepsilon\lambda y I\{|y| < 1\})\nu(dy)\} \\ &= \exp\{ia\varepsilon\lambda - i\lambda\varepsilon \int_{\mathbb{R}\setminus\{0\}} y I\{|y| < 1\}\nu(dy) - d\varepsilon^2\frac{\lambda^2}{2} + \int_{\mathbb{R}\setminus\{0\}} (e^{i\lambda y} - 1)\nu(d(\frac{y}{\varepsilon}))\} \\ &= \exp\{i\lambda a\varepsilon - i\lambda\varepsilon \int_{\mathbb{R}\setminus\{0\}} y I\{|y| < 1\}\nu(dy) + i\lambda \int_{\mathbb{R}\setminus\{0\}} y I\{|y| < 1\}\nu(d(\frac{y}{\varepsilon})) \\ &\quad - d\varepsilon^2\frac{\lambda^2}{2} + \int_{\mathbb{R}\setminus\{0\}} (e^{i\lambda y} - 1 - i\lambda y I\{|y| < 1\})\nu(d(\frac{y}{\varepsilon}))\}. \end{aligned}$$

Hence, $\varepsilon L(t)$ is a Lévy process with the generating triplet $(\varepsilon a - \varepsilon \int_{\mathbb{R}\setminus\{0\}} y I\{|y| < 1\}\nu(dy) + \int_{\mathbb{R}\setminus\{0\}} y I\{|y| < 1\}\nu(d(\frac{y}{\varepsilon})), d\varepsilon^2, \nu(d(\frac{y}{\varepsilon})))$. Using the equation (8), it is seen that the generator of εL_t is given in (9).

This completes the proof of Lemma 1. \square

Remark 1. The generator for the process X_t^ε in (1) is then

$$\begin{aligned} A^\varepsilon \varphi &= -U'(x)\varphi'(x) + \varepsilon a\varphi'(x) + \frac{1}{2}\varepsilon^2 d\varphi''(x) \\ &+ \int_{\mathbb{R} \setminus \{0\}} [\varphi(x + \varepsilon y) - \varphi(x) - \varepsilon I\{|y| < 1\} y\varphi'(x)] \nu(dy). \end{aligned} \quad (10)$$

We make the following assumptions for the SDE (1):

(A) There exists a function $g_1(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$ such that for any $\gamma > 0$ $\int_{\mathbb{R} \setminus [-\gamma, \gamma]} \nu(d(\frac{u}{\varepsilon})) \leq K(\gamma)g_1(\varepsilon)$ where $K(\gamma)$ is some function of γ .

(B) For any $\delta > 0$, there exists a positive constant $K < \infty$ such that $\int_{\mathbb{R} \setminus [-K, K]} \nu(d(\frac{u}{\varepsilon})) \leq \delta g_1(\varepsilon)$.

(C) There exists a measure $\nu^*(du)$ on $\mathbb{R} \setminus \{0\}$ such that $\frac{1}{g_1(\varepsilon)}\nu(d(\frac{u}{\varepsilon}))$ converges weakly to $\nu^*(du)$. The limit measure ν^* satisfies the condition that for any Borel set $A \subset \mathbb{R} \setminus \{0\}$ with measure 0, we have $\nu^*(A) = 0$.

(D) There exists a function $g_2(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$ and a positive constant $K < \infty$ such that

$$d\varepsilon^2 + \int_{\mathbb{R}} \frac{u^2}{1+u^2} \nu(d(\frac{u}{\varepsilon})) < Kg_2(\varepsilon).$$

And, there exists some $n > 0$ such that $(g_2(\varepsilon))^n \leq g_1(\varepsilon)$.

(E) There exists a $g_3(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$ and a positive constant $K < \infty$ such that

$$\int_{\mathbb{R}} \frac{u}{1+u^2} \nu(d(\frac{u}{\varepsilon})) < Kg_3(\varepsilon).$$

We consider a special class of symmetric Lévy measures on \mathbb{R} for $0 < \alpha < 2$:

$$\nu(du) = f(\ln|u|) \frac{du}{|u|^{1+\alpha}}, \quad (11)$$

where f is a nonnegative measurable function on \mathbb{R} such that this ν is a Lévy measure, i.e., it satisfies the above condition (2):

$$\int_{\mathbb{R} \setminus \{0\}} \frac{u^2}{1+u^2} f(\ln|u|) \frac{du}{|u|^{1+\alpha}} < \infty. \quad (12)$$

Being symmetric, this Lévy measure $\nu(du)$ automatically satisfies the condition (E).

Let $\delta > 0$. Define $G^\delta = \{x \in [-b, a] : \inf_{t \geq 0} \min(|Y(0, x, t) - a|, |Y(0, x, t) - (-b)|) \geq \delta\}$.

Theorem 1. Consider a class of symmetric Lévy measures on \mathbb{R} in (11). Assume that the conditions **(A)** – **(E)** be satisfied. Especially the condition **(C)** is satisfied, namely, there exists a positive function $\tilde{f}(\varepsilon)$ such that

$$\frac{f(\ln \frac{|u|}{\varepsilon})}{\tilde{f}(\varepsilon)} \frac{du}{|u|^{1+\alpha}} \rightharpoonup \nu^*(du).$$

weakly (as Borel measures on \mathbb{R}) to a certain measure ν^* . Then for any $x \in G^\delta$, we have

$$\lim_{\varepsilon \downarrow 0} \tilde{f}(\varepsilon) E_x \sigma(\varepsilon) = \frac{1}{\nu^*(\mathbb{R} \setminus [-b, a])},$$

or for $\varepsilon \downarrow 0$,

$$E_x \sigma(\varepsilon) \sim \frac{1}{\nu^*(\mathbb{R} \setminus [-b, a])} \frac{1}{\tilde{f}(\varepsilon)}.$$

Proof. Let us rewrite the generator for the one dimensional Markov process X_t^ε .

The generator A^ε depending on parameter ε for the one dimensional Markov process X_t^ε in equation (1) can be rewritten as in the following, using Lemma 1 and (10),

$$\begin{aligned} A^\varepsilon f(x) &= -U'(x)f'(x) + \varepsilon a f'(x) - \varepsilon \int_{\mathbb{R}} u I\{|u| < 1\} \nu(du) f'(x) + \frac{d\varepsilon^2}{2} f''(x) \\ &\quad + \int_{\mathbb{R}} [f(x+u) - f(x)] \nu(d(\frac{u}{\varepsilon})) \\ &= [-U'(x) + \varepsilon a - \varepsilon \int_{\mathbb{R}} u I\{|u| < 1\} \nu(du)] f'(x) + \int_{\mathbb{R}} \frac{u}{1+|u|^2} \nu(d(\frac{u}{\varepsilon})) f'(x) \\ &\quad + \frac{d\varepsilon^2}{2} f''(x) + \int_{\mathbb{R}} [f(x+u) - f(x) - \frac{u}{1+|u|^2} f'(x)] \nu(d(\frac{u}{\varepsilon})) \\ &= U^\varepsilon(x) f'(x) + \frac{d\varepsilon^2}{2} f''(x) + \int_{\mathbb{R}} [f(x+u) - f(x) - \frac{u}{1+|u|^2} f'(x)] \nu(d(\frac{u}{\varepsilon})). \end{aligned}$$

Here

$$U^\varepsilon(x) = -U'(x) + \varepsilon a - \varepsilon \int_{\mathbb{R}} u I\{|u| < 1\} \nu(du) + \int_{\mathbb{R}} (\frac{u}{1+|u|^2}) \nu(d(\frac{u}{\varepsilon})).$$

The result follows a similar idea in the proof of the Assertion in [6]. \square

3 Examples

We look at some applications of the above Theorem 1.

Example 1: α -stable symmetric Lévy Noise

This is a special case of Theorem 1 above with $f(\cdot) \equiv 1$. Consider X_t^ε defined in equation (1), where the Lévy process L_t is characterized by

$$Ee^{i\lambda L_t} = \exp\left\{-td\frac{\lambda^2}{2} + t \int_{\mathbb{R} \setminus \{0\}} (e^{i\lambda u} - 1 - i\lambda u I\{|u| < 1\}) \frac{1}{|u|^{1+\alpha}}(du)\right\}.$$

Here $d \geq 0$ and $0 < \alpha < 2$ are some constants. The Lévy jump measure is $\nu(du) = \frac{1}{|u|^{1+\alpha}}(du)$. This is a so called α -stable symmetric Lévy process, and it is heavy tailed and has infinite mass due to the strong intensity of small jumps.

Now, we try to verify the conditions in Theorem 1. Notice that $\nu(d(\frac{u}{\varepsilon})) = \frac{\varepsilon^\alpha}{|u|^{1+\alpha}}du$. It can be verified that there exist $g_1(\varepsilon) = g_2(\varepsilon) = g_3(\varepsilon) = \varepsilon^\alpha$, and $\nu^*(du) = \frac{1}{|u|^{1+\alpha}}du$ such that the conditions (A)–(D) are satisfied. Hence, for any $x \in G^\delta$, we have

$$\lim_{\varepsilon \downarrow 0} \varepsilon^\alpha E_x \sigma(\varepsilon) = \frac{1}{\int_a^\infty \frac{1}{|u|^{1+\alpha}}du + \int_{-\infty}^{-b} \frac{1}{|u|^{1+\alpha}}du} = \alpha \left[\frac{1}{a^\alpha} + \frac{1}{b^\alpha} \right]^{-1}.$$

Thus

$$E_x \sigma(\varepsilon) \sim \alpha \left[\frac{1}{a^\alpha} + \frac{1}{b^\alpha} \right]^{-1} \frac{1}{\varepsilon^\alpha}.$$

This is the result that Imkeller and Pavlyukevich [7] obtained earlier.

Example 2: A symmetric Lévy Noise

Consider X_t^ε defined in equation (1), with a special symmetric Lévy process L_t that is characterized by

$$Ee^{i\lambda L_t} = \exp\left\{-dt\frac{\lambda^2}{2} + t \int_{\mathbb{R} \setminus \{0\}} (e^{i\lambda u} - 1 - i\lambda u I\{|u| < 1\}) \nu(du)\right\}.$$

Here $d \geq 0$ and $\nu(du) = f(\ln|u|) \frac{du}{|u|^{1+\alpha}}$, $0 < \alpha < 2$ with $f(\ln|u|) = \frac{1}{|\ln|u||^{1+\alpha}}$. Such a $\nu(du)$ is a Levy measure satisfying the condition (2).

We claim that there exist $g_1(\varepsilon) = g_2(\varepsilon) = \frac{\varepsilon^\alpha}{-\ln \varepsilon}$ and $\nu^*(du) = \frac{1}{|u|^{1+\alpha}}du$ such that the conditions (A)–(D) are satisfied.

To verify condition (A), it is sufficient to show that for any $r > 0$ there exists some function of r , $K(r)$ that $\int_r^\infty \nu(d(\frac{u}{\varepsilon})) \leq K(r)(\frac{\varepsilon^\alpha}{-\ln \varepsilon})$ for ε small enough. We take a constant $C = |\min\{\ln r, 0\}|$. Notice that

$$\begin{aligned} \int_r^\infty \nu(d(\frac{u}{\varepsilon})) / (\frac{\varepsilon^\alpha}{-\ln \varepsilon}) &= \int_r^\infty \frac{1}{u^{1+\alpha} (|1 + \frac{\ln u}{-\ln \varepsilon}| + \frac{1}{-\ln \varepsilon})} du \\ &\leq \int_r^\infty \frac{1}{u^{1+\alpha} [1 - \frac{C}{-\ln \varepsilon} + \frac{1}{-\ln \varepsilon}]} du \\ &= \int_r^\infty \frac{1}{u^{1+\alpha} [1 + \frac{1-C}{-\ln \varepsilon}]} du = \frac{1}{\alpha [1 + \frac{1-C}{-\ln \varepsilon}]} r^\alpha \\ &\leq \frac{2}{\alpha} r^\alpha. \end{aligned}$$

for ε sufficiently small. To verify Condition **(B)**, we notice that for $K > 1$, $\int_K^\infty \nu(d(\frac{u}{\varepsilon})) / (\frac{\varepsilon^\alpha}{-\ln \varepsilon}) \leq \int_K^\infty \frac{1}{u^{1+\alpha}} du$ which is smaller than any $\delta > 0$, if K is big enough. To verify condition **(C)**, it is sufficient to show that for any $r > 0$,

$$\lim_{\varepsilon \downarrow 0} \int_r^\infty \frac{1}{\frac{\varepsilon^\alpha}{-\ln \varepsilon}} \nu(d(\frac{u}{\varepsilon})) = \int_r^\infty \frac{1}{u^{1+\alpha}} du.$$

This can be done by a Lebesgue convergence theorem. Finally, let us verify condition **(D)**. Since $\varepsilon^2 < \frac{\varepsilon^\alpha}{-\ln \varepsilon}$ for $0 < \varepsilon \ll 1$ and the measure ν is symmetric, we only need to show $\int_0^\infty \frac{u^2}{1+u^2} \nu(d(\frac{u}{\varepsilon})) / (\frac{\varepsilon^\alpha}{-\ln \varepsilon})$ is bounded by some constant K . Notice that

$$\begin{aligned} & \int_0^\infty \frac{u^2}{1+u^2} \nu(d(\frac{u}{\varepsilon})) / (\frac{\varepsilon^\alpha}{-\ln \varepsilon}) \\ &= \int_0^\infty (\frac{u^2}{1+u^2}) \frac{1}{u^{1+\alpha} [1 + \frac{\ln u}{-\ln \varepsilon} + \frac{1}{-\ln \varepsilon}]} (du) \\ &< \int_0^{\sqrt{\varepsilon}} \frac{u^{1-\alpha}}{|1 + \frac{\ln u}{-\ln \varepsilon} + \frac{1}{-\ln \varepsilon}|} du + \int_{\sqrt{\varepsilon}}^1 \frac{u^{1-\alpha}}{|1 + \frac{\ln u}{-\ln \varepsilon} + \frac{1}{-\ln \varepsilon}|} du + \int_1^\infty \frac{1}{u^{1+\alpha} [1 + \frac{\ln u}{-\ln \varepsilon} + \frac{1}{-\ln \varepsilon}]} du \end{aligned}$$

Here

$$\int_0^{\sqrt{\varepsilon}} \frac{u^{1-\alpha}}{|1 + \frac{\ln u}{-\ln \varepsilon} + \frac{1}{-\ln \varepsilon}|} du < \int_0^{\sqrt{\varepsilon}} \frac{u^{1-\alpha}}{\frac{1}{-\ln \varepsilon}} du = \frac{(-\ln \varepsilon)(\sqrt{\varepsilon})^{2-\alpha}}{2-\alpha} < K_1$$

for some constant K_1 if ε is sufficiently small. And,

$$\begin{aligned} & \int_{\sqrt{\varepsilon}}^1 \frac{u^{1-\alpha}}{|1 + \frac{\ln u}{-\ln \varepsilon} + \frac{1}{-\ln \varepsilon}|} du = \int_{\sqrt{\varepsilon}}^1 \frac{u^{1-\alpha}(-\ln \varepsilon)}{|\ln u - \ln \varepsilon| + 1} du \\ &< \int_{\sqrt{\varepsilon}}^1 \frac{u^{1-\alpha}(-\ln \varepsilon)}{\ln \sqrt{\varepsilon} - \ln \varepsilon + 1} du < \int_{\sqrt{\varepsilon}}^1 \frac{u^{1-\alpha}(-\ln \varepsilon)}{-\frac{1}{2} \ln \varepsilon} du \\ &= \frac{2}{2-\alpha} [1 - (\sqrt{\varepsilon})^{2-\alpha}] < K_2. \end{aligned}$$

for some constant K_2 if ε is small enough. Finally,

$$\int_1^\infty \frac{1}{u^{1+\alpha} [1 + \frac{\ln u}{-\ln \varepsilon} + \frac{1}{-\ln \varepsilon}]} du < \int_1^\infty \frac{1}{u^{1+\alpha}} du = \frac{1}{\alpha}.$$

Therefore, condition **(D)** is satisfied. By Theorem 1, we conclude that for any $x \in G^\delta$, we have

$$\lim_{\varepsilon \downarrow 0} \frac{\varepsilon^\alpha}{-\ln \varepsilon} E_x \sigma(\varepsilon) = \frac{1}{\int_a^\infty \frac{1}{|u|^{1+\alpha}} du + \int_{-\infty}^{-b} \frac{1}{|u|^{1+\alpha}} du} = \alpha \left[\frac{1}{a^\alpha} + \frac{1}{b^\alpha} \right]^{-1}.$$

So

$$E_x \sigma(\varepsilon) \sim \alpha \left[\frac{1}{a^\alpha} + \frac{1}{b^\alpha} \right]^{-1} \frac{|\ln(\varepsilon)|}{\varepsilon^\alpha}.$$

This mean exit time is asymptotically $O(\frac{|\ln \varepsilon|}{\varepsilon^\alpha})$. It is faster than exponential (the well-known Gaussian Brownian noise case [5]) but slower than polynomial (the stable Lévy noise case [7]; see also Example 1 above). Namely, for $0 < \varepsilon \ll 1$,

$$O(\frac{1}{\varepsilon^\alpha}) < O(\frac{|\ln \varepsilon|}{\varepsilon^\alpha}) < \exp(\frac{C}{\varepsilon^2}). \quad (13)$$

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